Interest Rates and Financial Equivalence

Topic 3

3.1

Interest: the cost of money

- Where have you heard of interest before?

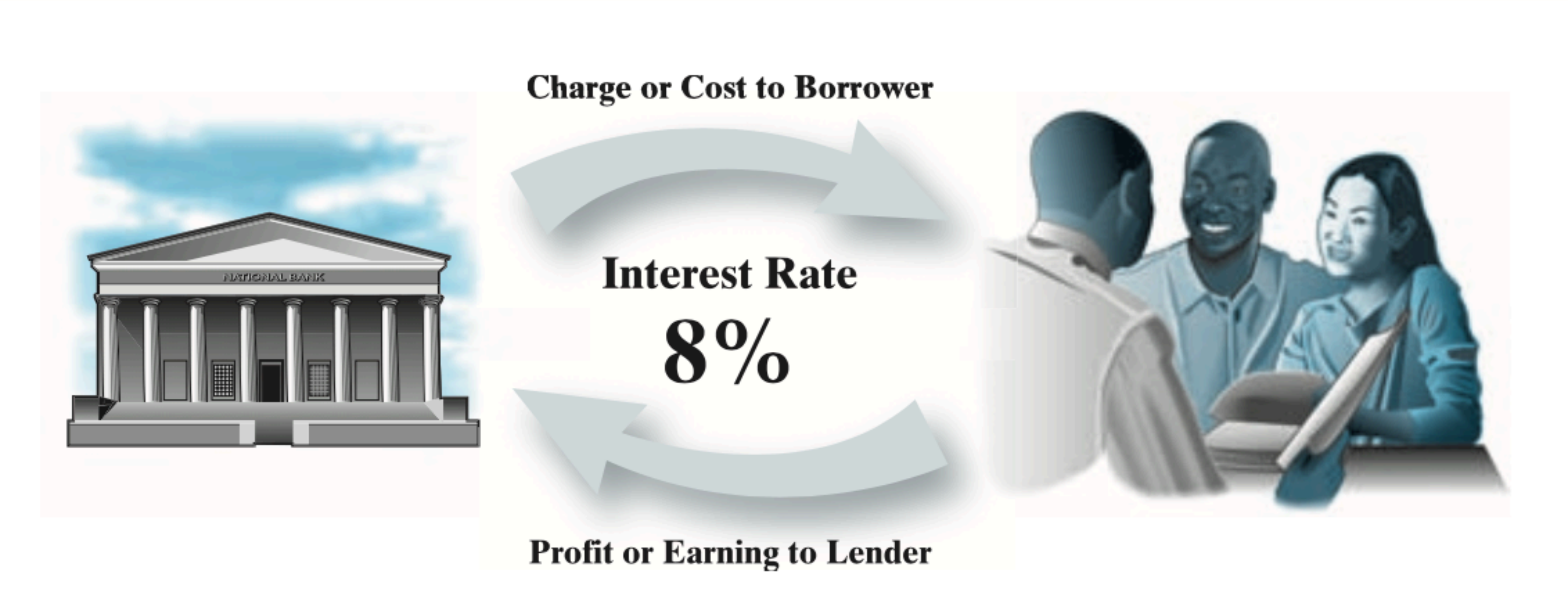
- Money is a commodity, and like other goods that are bought and sold, money costs money.

How much does money cost?

- established and measure by a “market interest rate”
  - a percentage that is periodically applied and added to an amount of money over a specific length of time

- When money is borrowed, the interest paid is that charge to the borrower for the use of the lender’s property.
- When money is lent or invested, the interest earned is the lender’s gain from providing a good to another.

- Interest: “the cost of having money available for use”


Charge or Cost to Borrower
NATIONAL BANK
Interest Rate
8%
Profit or Earning to Lender

The Time Value of Money

- Should I buy something now?
  - I really want a new watch (500 SAR)
  - I only have 500 SAR

  - Case 1:
    - Invest (6% interest) → in one year I have 530 SAR
    - Inflation (4%) → in one year watch costs 520 SAR (10 SAR extra)
  - Case 2:
    - Invest (6% interest) → in one year I have 530 SAR
    - Inflation (8%) → in one year watch costs 540 SAR (not enough money)

Time is a resource

- Then we need to budget for time too!
- \$1,000,000 USD at a 10% interest rate earns \$100,000/year
  - Waiting to receive \$1,000,000 involves a significant sacrifice!
- Money has ”earning power” and “purchasing power”
- Purchasing power is the value of a currency expressed in terms of the amount of goods or services that one unit of money can buy.

A note: (skip in class; students read later)

- if we want to know the true desired earnings in isolation from inflation, we can determine the real interest rate.
- The earning power of money and its loss of value because of inflation are calculated by different analytical techniques.
- The interest rate we will generally talk about is the “market interest rate,” which takes into account earning power, as well as inflation.

Elements of Transactions Involving Interest

- An initial amount of money in transactions involving debt or investments is called the **principal**.
- The **interest rate** measure the cost or price of money and is expressed as a percentage per period of time.
- A period of time, called the **interest period**, determines how frequently interest is calculated.
- A specified length of time marks the duration of the transaction and thereby establishes a certain **number of interest periods**
- A **plan for receipts or disbursements** yields a particular cash flow pattern over a specified length of time.
- A **future amount of money** results from the cumulative effects of the interest rate over a number of interest periods.

A_n = A discrete payment or receipt occurring at the end of some interest period.
i = The interest rate per interest period.
N = The total number of interest periods.
P = A sum of money at a time chosen as time zero for purposes of analysis; sometimes referred to as the **present value** or **present worth**.
F = A future sum of money at the end of the analysis period. This sum may be specified as F_N.
A = An end-of-period payment or receipt in a uniform series that continues for N periods. This is a special situation where $A_1 = A_2 = \cdots = A_N$.
V_n = An equivalent sum of money at the end of a specified period n that considers the effect of the time value of money. Note that $V_0 = P$ and $V_N = F$.

Example Interest Transaction

- A medical device manufacturing company buys a machine for $25,000 and borrows $20,000 from a bank at 9% annual interest rate.
- The company pays a $200 loan origination fee when the loan begins.
- The bank offers 2 payment plans, one with equal payments over 5 years, the other with a single payment made after 5 years only.

		Payments	
End of Year	**Receipts**	**Plan 1**	**Plan 2**
Year 0	$20,000.00	$ 200.00	$ 200.00
Year 1		5,141.85	0
Year 2		5,141.85	0
Year 3		5,141.85	0
Year 4		5,141.85	0
Year 5		5,141.85	30,772.48

$P = \$20{,}000,\ A = \$5{,}141.85,\ F = \$30{,}772.48$

Note: You actually borrow $19,800 with the origination fee of $200, but you pay back on the basis of $20,000.

End of Year	Receipts	Payments	
		Plan 1	Plan 2
Year 0	$20,000.00	$ 200.00	$ 200.00
Year 1		5,141.85	0
Year 2		5,141.85	0
Year 3		5,141.85	0
Year 4		5,141.85	0
Year 5		5,141.85	30,772.48

$P = \$20{,}000$, $A = \$5{,}141.85$, $F = \$30{,}772.48$

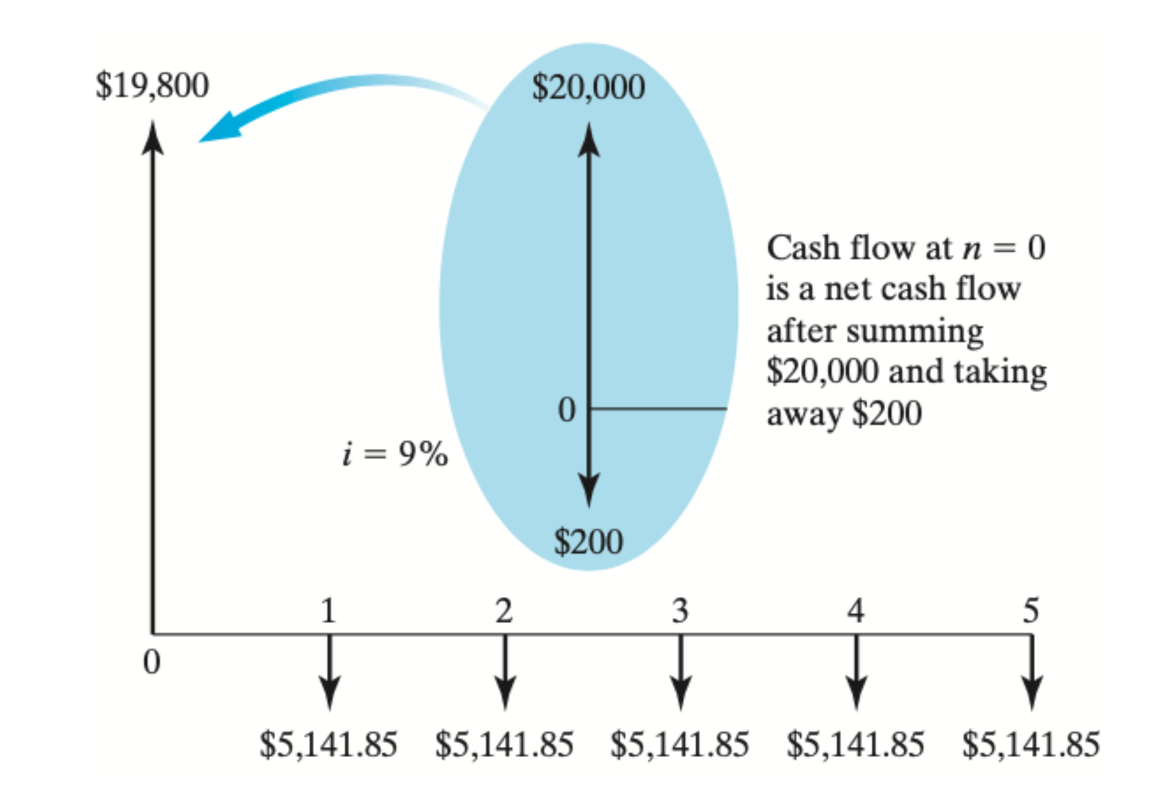
Note: You actually borrow $19,800 with the origination fee of $200, but you pay back on the basis of $20,000.

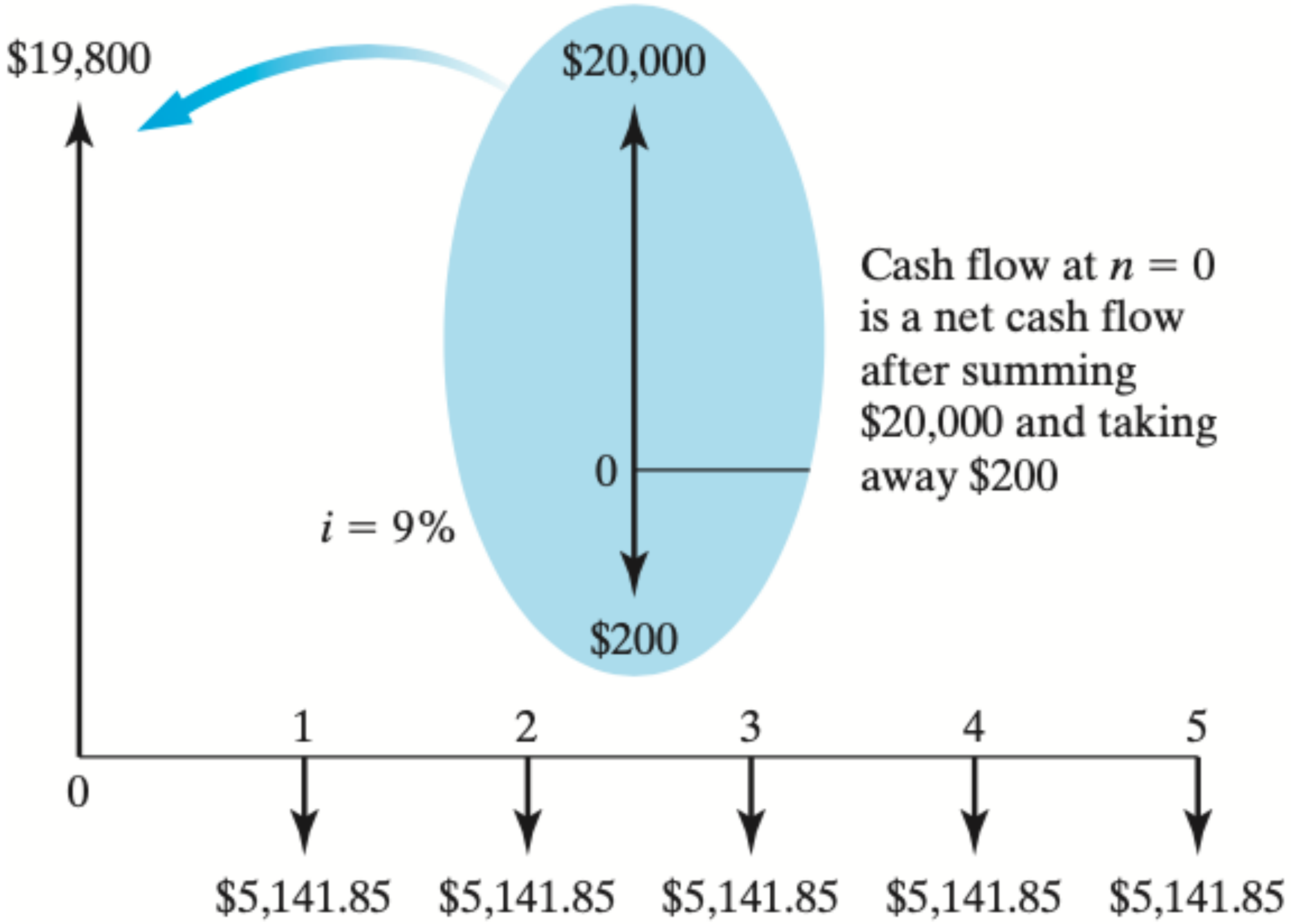
Plan 1: principal amount, P, is $20,000, interest rate, i, is 9%. Interest period is 1 year. duration of the transaction is 5 years. (5 interest periods: N=5) Cash flow pattern: 5 equal payments, A = $5,141.85.

Plan 2: "grace period" + single payment F = $30,772.78

Cash Flow Diagram

- Represent time by a horizontal line marked off with the number of interest periods specified.
- Upward arrows = positive flows (receipts)
- Downward arrows = negative flows (disbursements)
- Arrows represent "net cash flows" (2 or more are summed together)




$19,800
$20,000
Cash flow at n = 0
is a net cash flow
after summing
$20,000 and taking
away $200
0
i = 9%
$200
1
2
3
4
5
0
$5,141.85
$5,141.85
$5,141.85
$5,141.85
$5,141.85

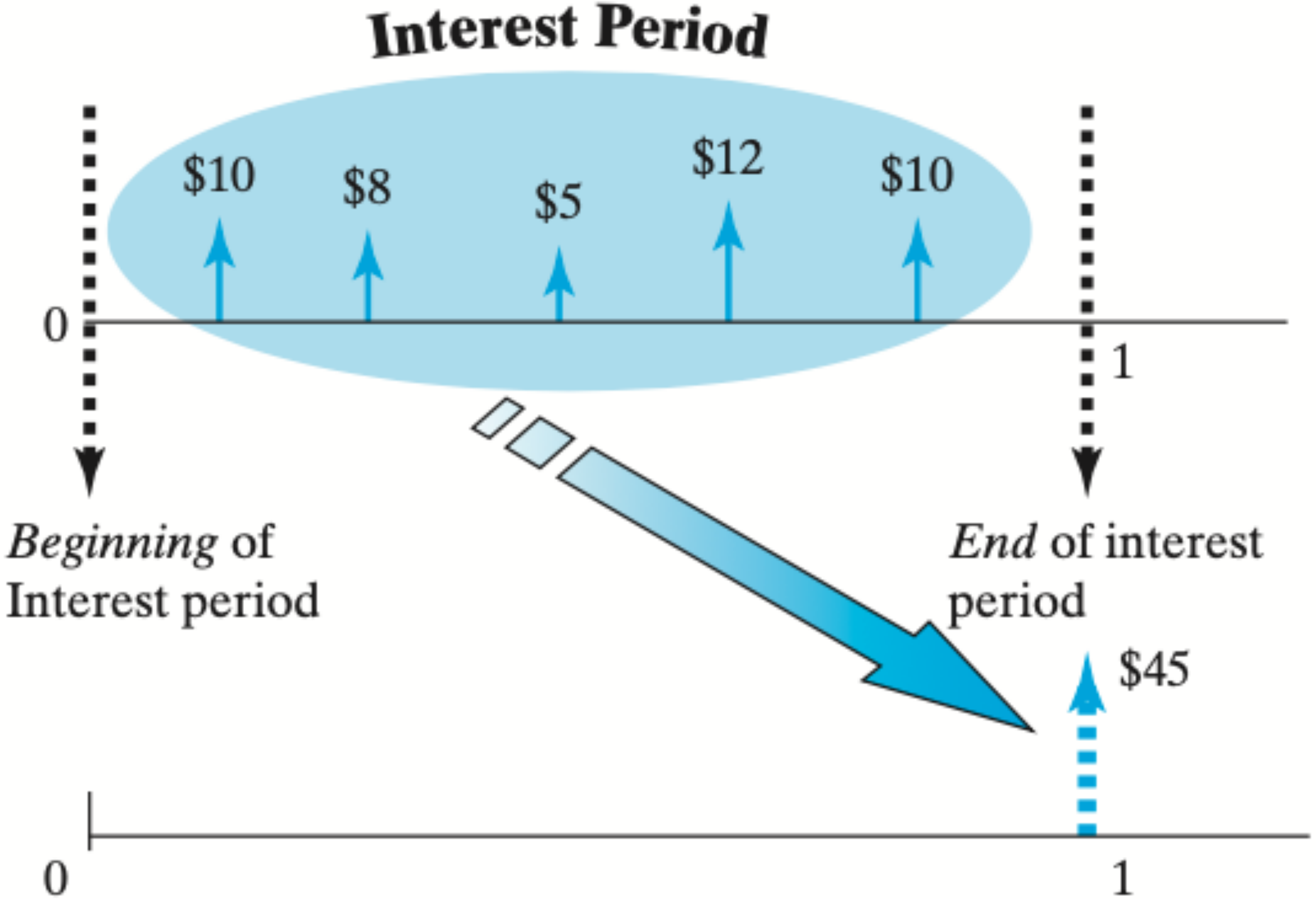


Figure 3.5 Any cash flows occurring during the interest period are summed to a single amount and placed at the end of the interest period.

Methods of Calculating Interest

- **Simple Interest**: the interest rate is applied only to the original principal amount in computing the amount of interest.
  - Any interest earned during each interest period does not earn additional interest, even if you don't withdraw if.

$$I = (iP)N.$$

$$F = P + I = P(1 + iN).$$

  - Commonly used with add-on loans or bonds
- **Compound Interest**: (used most of the time, almost exclusively by engineering economic analysis)
  - interest earned in each period is calculated on the basis of the total amount at the end of the previous period.
  - total includes original + accumulated interest left in account

Methods of Calculating Interest

- **Simple Interest**: the interest rate is applied only to the original principal amount in computing the amount of interest.
  - Any interest earned during each interest period does not earn additional interest, even if you don't withdraw if.

$$I = (iP)N.$$

$$F = P + I = P(1 + iN).$$

  - Commonly used with add-on loans or bonds
- **Compound Interest**: (used most of the time, almost exclusively by engineering economic analysis)
  - interest earned in each period is calculated on the basis of the total amount at the end of the previous period.
  - total includes original + accumulated interest left in account

$$F = P(1 + i)^N.$$

Work alone:

Suppose you deposit $1,000 in a bank savings account that pays interest at a rate of 10% compounded annually. Assume that you don’t withdraw the interest earned at the end of each period (one year), but let it accumulate. How much would you have at the end of year 3?

SOLUTION

Given: $P = \$1{,}000$, $N = 3$ years, and $i = 10\%$ per year.
Find: F.

Applying Eq. (3.3) to our three-year, 10% case, we obtain

$$F = \$1{,}000(1 + 0.10)^3 = \$1{,}331.$$

The total interest earned is \$331, which is \$31 more than was accumulated under the simple-interest method (Figure 3.6). We can keep track of the interest accruing process more precisely as follows:

Period	Amount at Beginning of Interest Period	Interest Earned for Period	Amount at End of Interest Period
1	\$1,000	\$1,000(0.10)	\$1,100
2	1,100	1,100(0.10)	1,210
3	1,210	1,210(0.10)	1,331

Problem

- What is the difference in the final amount, F_1 and F_2, if you (1) invest 10,000 SAR with a 10% annual interest rate (compound interest) and (2) invest 10,000 SAR with a 10% annual interest rate (simple interest)?
- Write your answer in terms of N, the number of periods/years.

Problem

- What is the difference in the final amount, F_1 and F_2, if you (1) invest 10,000 SAR with a 10% annual interest rate (compound interest) and (2) invest 10,000 SAR with a 10% annual interest rate (simple interest)?
- Write your answer in terms of N, the number of periods/years.
- Draw a cash flow diagram for the investment above.

Problem

- Considering compound interest, derive the equation for total interest earned (I) over N periods.
- Write your answer in terms of P, i, and N.

Problem

Derive the additional interest earned with compound interest compared to simple interest. Write your answer in terms of P, i, and N.

What is the effect of i and N on the difference?

What is the difference when $N = 1$?

3.2

Economic Equivalence

Question

- If receiving $100 today is not the same thing as receiving $100 at any future point, how do we measure and compare various cash flows?

- $20,000 today?
- $50,000 10 years from now?
- $8,000 for each year for the next 10 years

- Today we'll discuss techniques for making this decision.

The Central Question

- To decide between alternative cash flows, we should compare their "economic worth."
- It would be simple if we didn't have to compare the time value of money (just add the values)
- Because money has time-value, we need to consider:
  - the magnitude of the payment
  - the direction of the payment (receipt or disbursement)
  - the timing of the payment: when is it made?
  - the interest rate in operation during the period

Economic Equivalence

- The process of comparing two different cash amounts at different points in time
- Economic equivalence exists between cash flows that have the same economic effect and could therefore be traded for one another in the financial marketplace (assumed to exist).
- Any cash flow could be converted to an equivalent cash flow at any point in time.
  - We could find the equivalent future value F of a present amount P at an interest rate i, at period n.
  - OR determine equivalent present value P of N equal payments A.
  - OR compare two cash flows by converting them to the same point in time

Equivalence

- If two financial proposals that appear to be quite different turn out to have the same monetary value, then we can be *economically* indifferent to choosing between them.

Compare payment plans

TABLE 3.2 **Typical Repayment Plans for a Bank Loan of $20,000 (for N = 5 years and i = 9%)**

	Repayments		
	Plan 1	**Plan 2**	**Plan 3**
Year 1	$ 5,141.85	0	$ 1,800.00
Year 2	5,141.85	0	1,800.00
Year 3	5,141.85	0	1,800.00
Year 4	5,141.85	0	1,800.00
Year 5	5,141.85	$30,772.48	21,800.00
Total of payments	$25,709.25	$30,772.48	$29,000.00
Total interest paid	$ 5,709.25	$10,772.48	$ 9,000.00

Plan 1: Equal annual installments; Plan 2: End-of-loan-period repayment of principal and interest; Plan 3: Annual repayment of interest and end-of-loan repayment of principal

Compare payment plans

TABLE 3.2 **Typical Repayment Plans for a Bank Loan of $20,000 (for $N = 5$ years and $i = 9\%$)**

	Repayments		
	Plan 1	Plan 2	Plan 3
Year 1	$ 5,141.85	0	$ 1,800.00
Year 2	5,141.85	0	1,800.00
Year 3	5,141.85	0	1,800.00
Year 4	5,141.85	0	1,800.00
Year 5	5,141.85	$30,772.48	21,800.00
Total of payments	$25,709.25	$30,772.48	$29,000.00
Total interest paid	$ 5,709.25	$10,772.48	$ 9,000.00

These plans are actually equivalent, and the bank doesn't care which one you choose.

Plan 1: Equal annual installments; Plan 2: End-of-loan-period repayment of principal and interest; Plan 3: Annual repayment of interest and end-of-loan repayment of principal

Equivalence Calculations

- An application of the compound-interest relationship from last lecture.

- 1,000 SAR invested at 12$ interest rate for 5 years:

$$F = P(1+i)^N$$

$$F = 1{,}000(1+0.12)^5 = 1{,}762.34$$

- The cash flow above is equivalent to receiving $1,762.34 in five years.

Example

Suppose you are offered the alternative of receiving either \$3,000 at the end of five years or P dollars today. There is no question that the \$3,000 will be paid in full (no risk). Because you have no current need for the money, you would deposit the P dollars in an account that pays 8% interest. What value of P would make you indifferent to your choice between P dollars today and the promise of \$3,000 at the end of five years?

Strategy

- Our job is to determine the present amount that is economically equivalent to $3,000 in five years, given the investment potential of 8% per year. Note that the statement of the problem assumes that you would exercise the option of using the earning power of your money by depositing it. The "indifference" ascribed to you refers to economic indifference; that is, in a marketplace where 8% is the applicable interest rate, you could trade one cash flow for the other.

Solution

Given: $F = \$3{,}000$, $N = 5$ years, and $i = 8\%$ per year.

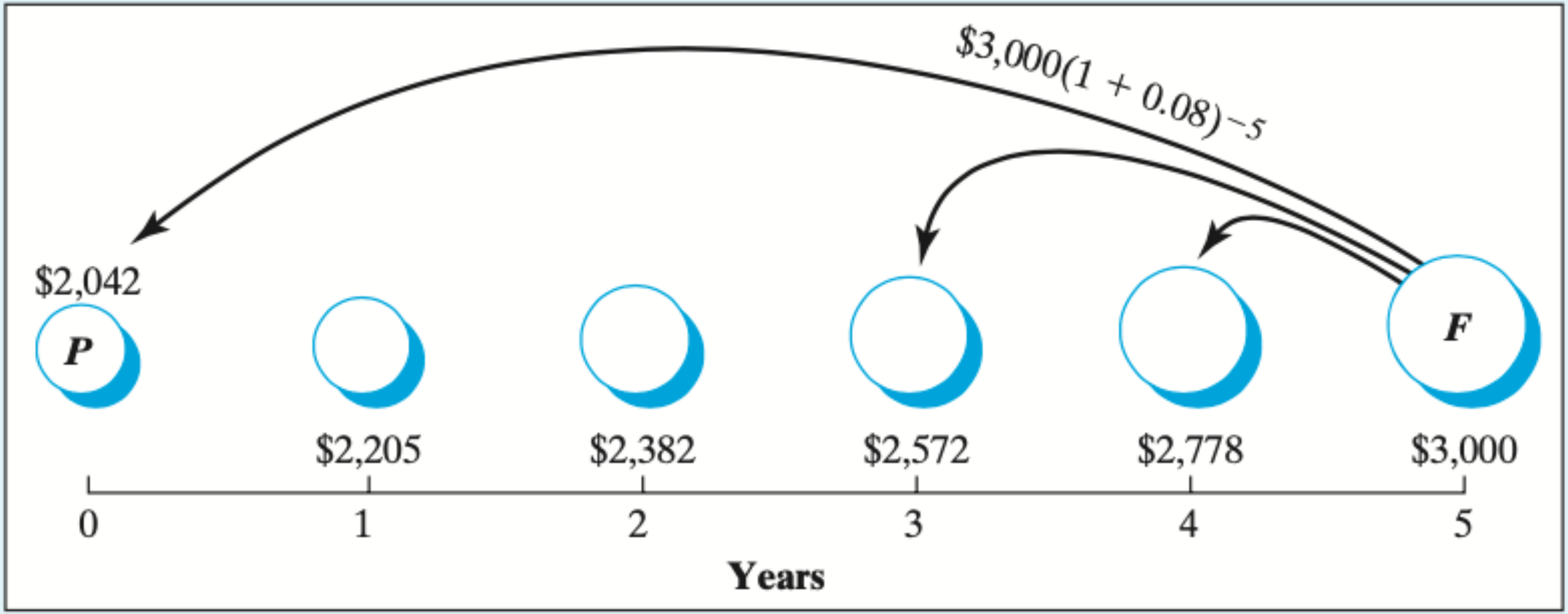
Find: P.

Equation: Eq. (3.3), $F = P(1 + i)^N$.
Rearranging terms to solve for P gives

$$P = \frac{F}{(1 + i)^N}.$$

Substituting yields

$$P = \frac{\$3{,}000}{(1 + 0.08)^5} = \$2{,}042.$$


$3,000(1 + 0.08)−5
$2,042
P
F
$2,205
$2,382
$2,572
$2,778
$3,000
0
1
2
3
4
5
Years

General Principles

1. Equivalence calculations made to compare alternatives require a common time basis (common base period).

- This is like converting fractions to a common denominator to compare their values.
- We could choose any time point! (present, future) Choose whatever is convenient.

Example

- Are the two investments from the previous problem also equivalent at year 3?
  - i = 8%, P = $2,042
  - $3,000 in 5 years.

Solution

SOLUTION

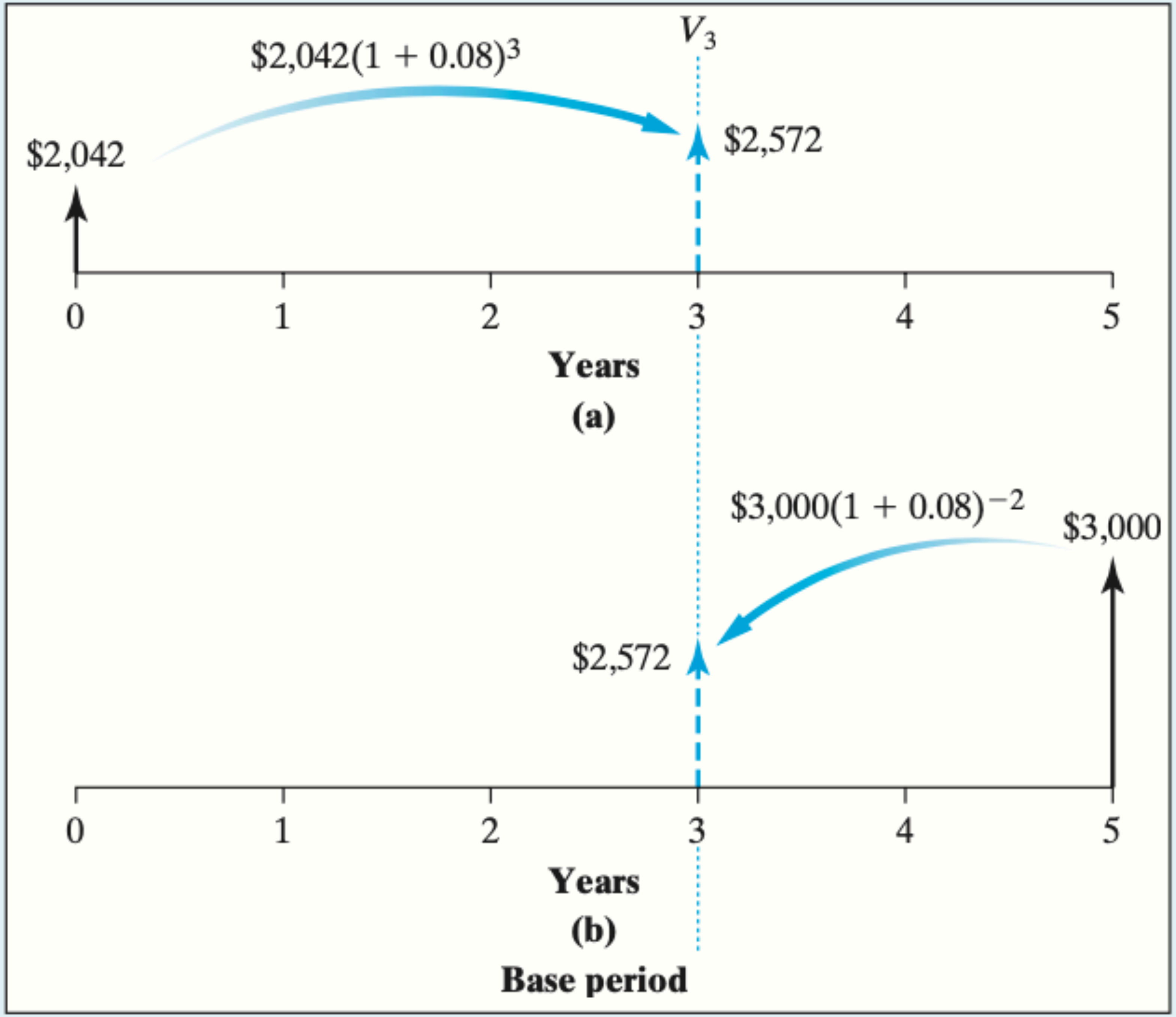
Given:

(a) $P = \$2{,}042; \quad i = 8\%$ per year; $\quad N = 3$ years.
(b) $F = \$3{,}000; \quad i = 8\%$ per year; $\quad N = 5 - 3 = 2$ years.

Find: (1) V_3 for part (a); (2) V_3 for part (b). (3) Are these two values equivalent?

Equation:

(a) $F = P(1 + i)^N$.
(b) $P = F(1 + i)^{-N}$.



V_3
$2,042(1 + 0.08)^3
$2,042
$2,572
0
1
2
3
4
5
Years
(a)
$3,000(1 + 0.08)^{−2}
$3,000
$2,572
0
1
2
3
4
5
Years
(b)
Base period

1. The equivalent worth of $2,042 after three years is

$$\begin{aligned} V_3 &= 2{,}042(1 + 0.08)^3 \\ &= \$2{,}572. \end{aligned}$$

2. The equivalent worth of the sum $3,000 two years earlier is

$$\begin{aligned} V_3 &= F(1 + i)^{-N} \\ &= \$3{,}000(1 + 0.08)^{-2} \\ &= \$2{,}572. \end{aligned}$$

General Principles

2. Equivalence depends on interest rate

- Any change in interest rate affects the calculation and thus the equivalence.

General Principles

3. Equivalence calculations may require the conversion of multiple payment cash flows to a single cash flow

- Part of the task of comparing cash flow series involves moving each individual cash flow in the series to the same single point in time and summing these values to yield a single equivalent cash flow.

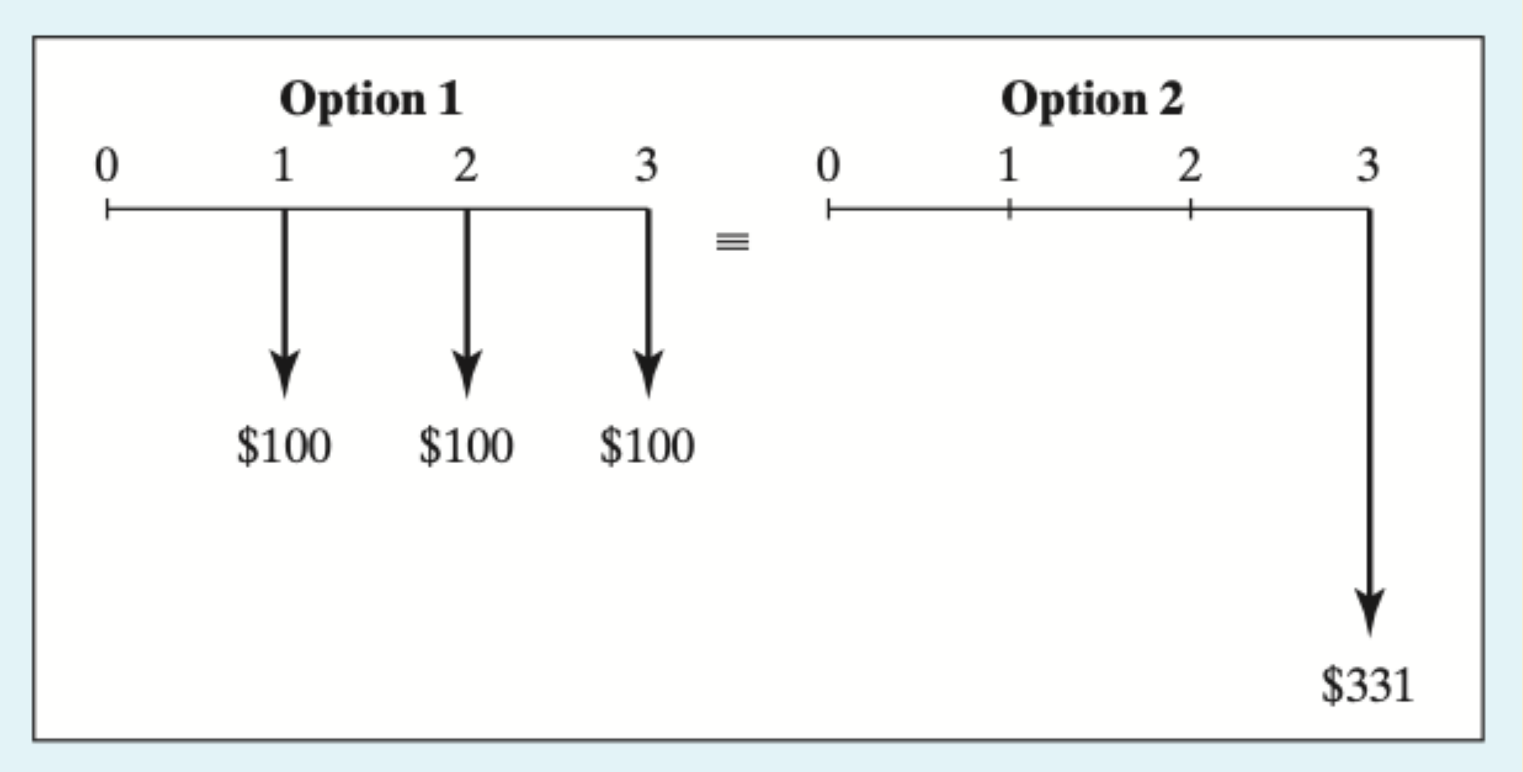
Example

- Loan: $1,000 from a bank for 3 years, i = 10% annual interest
- Two plans:
  - repaying the interest charges for each year at the end of that year and repaying the principal at the end of year 3
  - repaying the load all at once (interest and principal) at the end of year 3
- Determine whether these 2 options are equivalent.

Options	Year 1	Year 2	Year 3
• Option 1: End-of-year repayment of interest, and principal repayment at end of loan	$100	$100	$1,100
• Option 2: One end-of-loan repayment of both principal and interest	0	0	1,331

Strategy

- Remember that “<u>the only thing that matters is the difference between alternatives</u>!”
- Both plans pay principal at year 3 ($1,000), so this amount can be removed.
- Becomes:



Option 1 Equivalence Calculations

$$F_3 \text{ for } \$100 \text{ at } n = 1: \$100(1 + .10)^{3-1} = \$121;$$

$$F_3 \text{ for } \$100 \text{ at } n = 2: \$100(1 + .10)^{3-2} = \$110;$$

$$F_3 \text{ for } \$100 \text{ at } n = 3: \$100(1 + .10)^{3-3} = \underline{\$100};$$

$$\text{Total} = \$331.$$

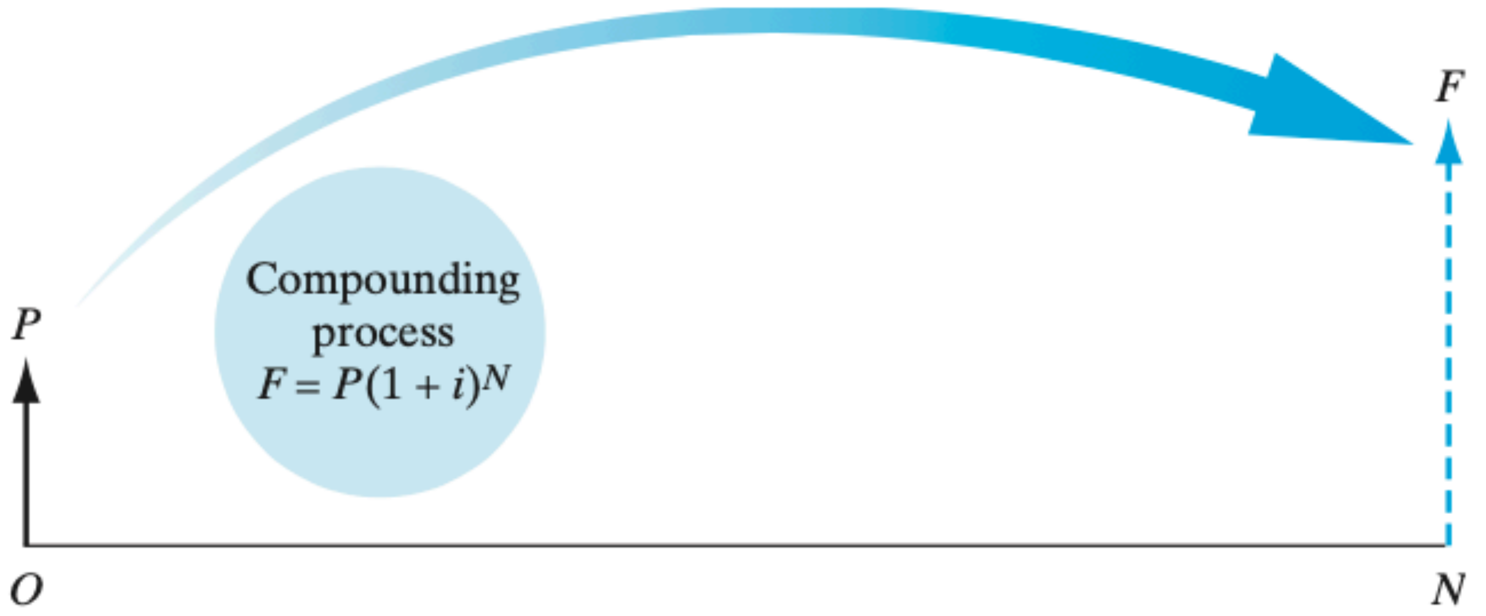
= Option 2 @ year 3.

General Principles

4. Equivalence is maintained regardless of point of view

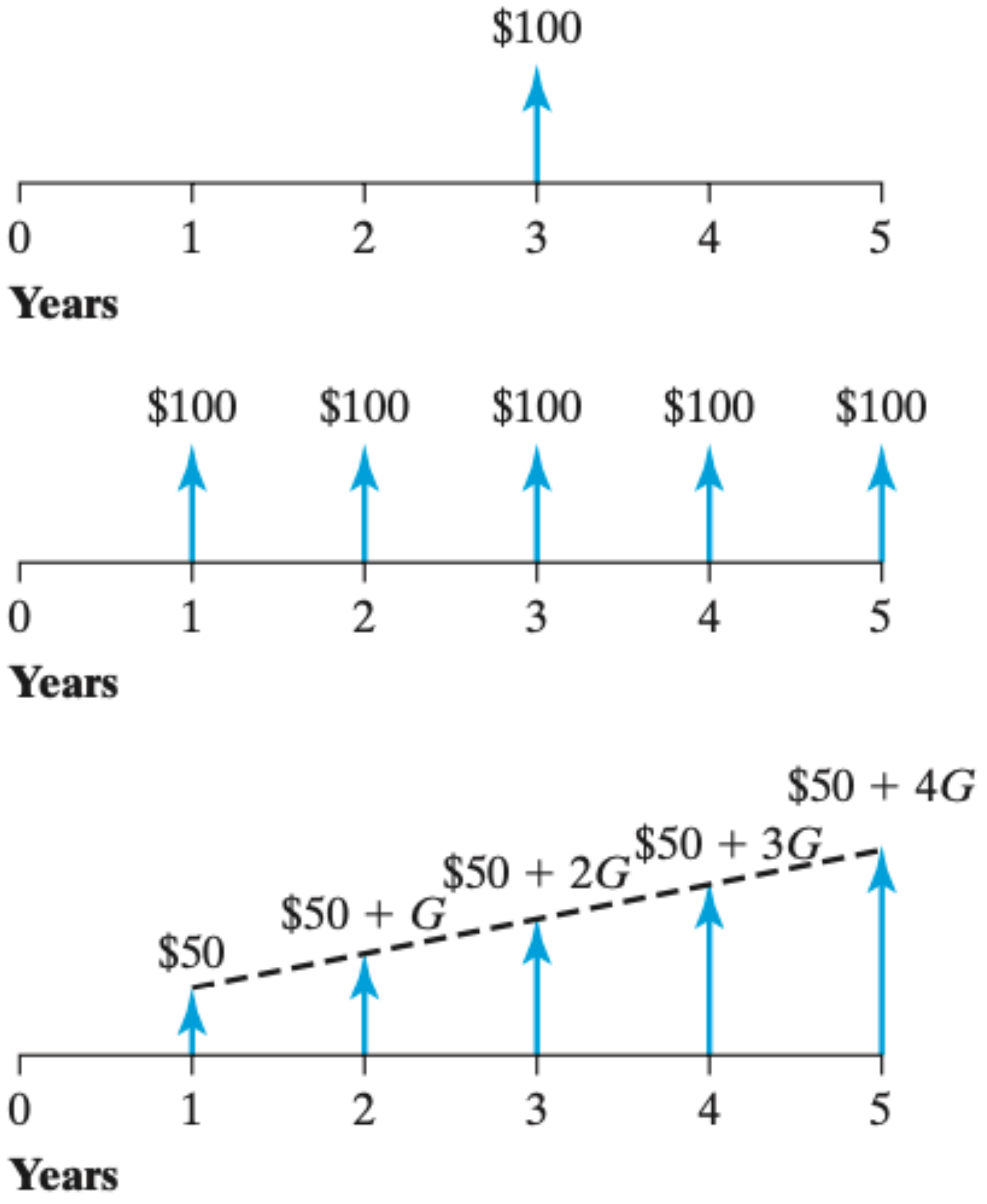
- If they are equivalent for bank, they’re equivalent for borrower also.

Compounding process: the process of computing the future value of a current sum.



Discounting process: the process of calculating the present value of a future amount.

Types of Cash Flows

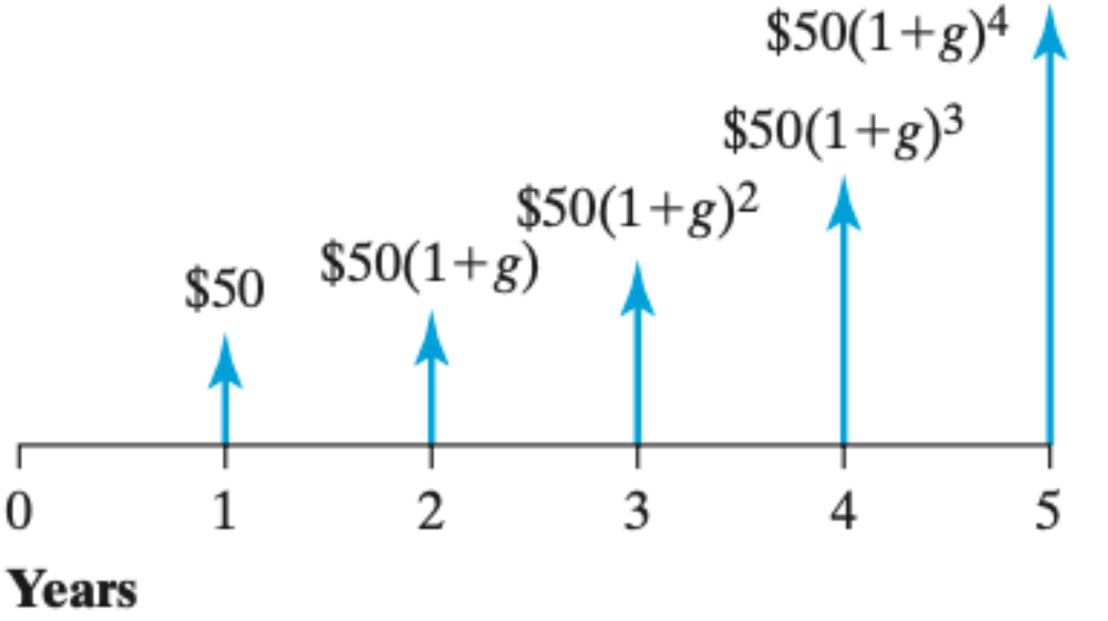


(a) Single cash flow

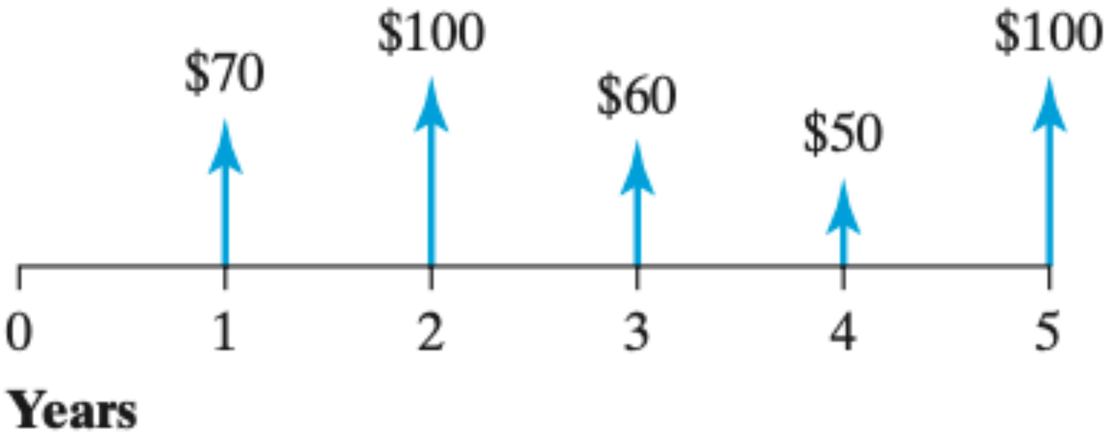
(b) Equal (uniform) payment series at regular intervals

(c) Linear gradient series, where each cash flow in the series increases or decreases by a fixed amount *G*

Types of Cash Flows



(d) Geometric gradient series, where each cash flow in the series increases or decreases by a fixed rate (percentage) *g*



(e) Irregular payment series, which exhibits no regular overall pattern

Using Tables

What is F for P = $20,000, 12% annual interest, 15 years

$$F = \$20{,}000(1 + 0.12)^{15}$$

Using Tables

What is F for P = $20,000, 12% annual interest, 15 years

$$F = \$20{,}000(1 + 0.12)^{15}$$

$$F = \$20{,}000\underbrace{(1 + 0.12)^{15}}_{5.4736} = \$109{,}472.$$

This is the <u>single-payment compound-amount factor</u>

Using Tables

What is F for P = \$20,000, 12% annual interest, 15 years

$$F = \$20{,}000(1 + 0.12)^{15}$$

$$F = \$20{,}000\underbrace{(1 + 0.12)^{15}}_{5.4736} = \$109{,}472.$$

This is the single-payment compound-amount factor

$$F = P(1 + i)^N = P(F/P, i, N)$$

12.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.1200	0.8929	1.0000	1.0000	0.8929	1.1200	0.0000	0.0000	1
2	1.2544	0.7972	2.1200	0.4717	1.6901	0.5917	0.4717	0.7972	2
3	1.4049	0.7118	3.3744	0.2963	2.4018	0.4163	0.9246	2.2208	3
4	1.5735	0.6355	4.7793	0.2092	3.0373	0.3292	1.3589	4.1273	4
5	1.7623	0.5674	6.3528	0.1574	3.6048	0.2774	1.7746	6.3970	5
6	1.9738	0.5066	8.1152	0.1232	4.1114	0.2432	2.1720	8.9302	6
7	2.2107	0.4523	10.0890	0.0991	4.5638	0.2191	2.5515	11.6443	7
8	2.4760	0.4039	12.2997	0.0813	4.9676	0.2013	2.9131	14.4714	8
9	2.7731	0.3606	14.7757	0.0677	5.3282	0.1877	3.2574	17.3563	9
10	3.1058	0.3220	17.5487	0.0570	5.6502	0.1770	3.5847	20.2541	10
11	3.4785	0.2875	20.6546	0.0484	5.9377	0.1684	3.8953	23.1288	11
12	3.8960	0.2567	24.1331	0.0414	6.1944	0.1614	4.1897	25.9523	12
13	4.3635	0.2292	28.0291	0.0357	6.4235	0.1557	4.4683	28.7024	13
14	4.8871	0.2046	32.3926	0.0309	6.6282	0.1509	4.7317	31.3624	14
15	5.4736	0.1827	37.2797	0.0268	6.8109	0.1468	4.9803	33.9202	15